## 1 Introduction -Great Oak Mu Alpha Theta

Skip the following section if you already understand the fundamentals of roots, polynomials, or if you have already completed Algebra 2.

The root of a polynomial is the value at which the function evaluates to zero. Roots are also called "zeros," and even "solutions" to the given equation. In example, the below equation has roots of $x=2$ and $x=-2$.
$x^{2}-4=0$
Roots are useful in determining the nature of polynomials, from end behaviours to solutions of physical systems. A polynomial can be expressed in the following forms:
$f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+a_{n-2} x^{n-2}+\ldots+a_{2} x^{2}+a_{1} x+a_{0}$
The roots of a quadratic in standard form can be expressed as:
$x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$
There is a root of multiplicity 2 if $b^{2}=4 a c$
Further readings include the Fundamental Theorem of Algebra, Descartes's Rule of signs, and the Rational Root Theorem.

## 2 Formula Sheet

Here is a list of formulas you may utilize when analyzing the roots of a given polynomial. They are collectively called Vieta's formulas, and enable the solutions to various desired quantities without necessitating factorization of the polynomial, or other nonanalytical algebraic methods. You should memorize them! (Useful for SAT, ACT, and more importantly math club).

For a quadratic, the roots $r_{1, r} 2$ for form of $a x^{2}+b x+c$ hold true for the following:
$r_{1}+r_{2}=\frac{-b}{a}$
$r_{1} r_{2}=\frac{c}{a}$
(The following is taken from the AOPS Wikipedia.) For a polynomial of the form $f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\ldots+a_{1} x+a_{0}$ with roots $r_{1}, r_{2}, r_{3}, \ldots r_{n}$, Vieta's formulas state that:

$$
\begin{aligned}
& \quad \mathrm{s}_{1}=r_{1}+r_{2}+r_{3}+\cdots+r_{n}=-\frac{a_{n-1}}{a_{n}} \\
& s_{2}=r_{1} r_{2}+r_{1} r_{3}+r_{1} r_{4}+\cdots+r_{n-2} r_{n-1}=\frac{a_{n-2}}{a_{n}} \\
& s_{3}=r_{1} r_{2} r_{3}+r_{1} r_{2} r_{4}+\cdots+r_{n-2} r_{n-1} r_{n}=-\frac{a_{n-3}}{a_{n}} \\
& \vdots \\
& s_{n}=r_{1} r_{2} r_{3} \cdots r_{n}=(-1)^{n} \frac{a_{0}}{a_{n}} .
\end{aligned}
$$

These formulas are widely used in competitions, and it is best to remember that when the $n$ roots are taken in groups of $k$ (i.e. $r_{1}+r_{2}+r_{3} \ldots+r_{n}$ is taken in groups of 1 and $r_{1} r_{2} r_{3} \ldots r_{n}$ is taken in groups of $n$ ), this is equivalent to $(-1)^{k} \frac{a_{n-k}}{a_{n}}$. Simply put, use $s_{1}$ for the summation of the roots and $s_{n}$ for the product of roots. Use $s_{2}, s_{3}$ etc, if desiring the "pairwise" product of the roots. In example, use $s_{2}$ if wanting to compute " $x y+x z+y z$."

## 3 Problems Level 1

1.0 Using the quadratic formula, express the sum of the roots for the given polynomial.
(The answer has 5 characters if you were to type it!)

$$
a x^{2}+b x+c=0
$$

2.0 Find the product of the roots for polynomial 1 . Divide that quantity by the smallest root of polynomial 2 .
(1.0) $x^{2}-11 x+24$
(2.0) $2 x^{2}+8 x+3 x+12=0$
3.0 If $x_{1}, x_{2}$ are the roots of the equation $x^{2}+11 x+12=0$, determine the value of $x_{1}^{2}+x_{2}^{2}$
4.0 Find the product of the roots that satisfy the below equation.

$$
\left(8 x^{3}+4 x^{2}+743 x+30\right)\left(3 x^{4}-68 x^{2}+20\right)=0
$$

4.5 Find the sum of the distinct roots for the quadratic. $x^{2}-84 x+1764$
4.6 HMMT 1998, Two roots of $x^{4}+a x^{2}+b x+c=0$ are known by inspection. They are 2, and -3 . Mr. Maxey whispers under his breath that the third known root is the range of the prior known roots. Compute the value of $\mathrm{a}+\mathrm{b}+\mathrm{c}$ for the polynomial. All roots for this polynomial are real.

## 4 Problems Level 2

5.0 Henry wants to find the roots of this given polynomial to work on his mousetrap vehicle. Let the roots of his polynomial be $r_{1}, r_{2}, r_{3}$ for polynomial $5 x^{3}-11 x^{2}+7 x+3$. Evaluate $r_{1}^{2}+r_{2}^{2}+r_{3}^{2}$ to find his vehicle's needed torque for acceleration.
6.0 Let $a$ and $b$ be the roots of the equation $x^{2}-m x+2=0$. Suppose that $a+\frac{1}{b}$ and $b+\frac{1}{a}$ are the roots of the equation $x^{2}-p x+q=0$. What is $q$ ?
6.6 Professor Axes has a rectangular box in which he needs to know the length of the 3 -dimensional diagonal to place string for his cat trap. He knows the roots of the given polynomial, $\mathrm{r}, \mathrm{s}$, and t which have units of meters, satisfy the lengths of his rectangular box. Compute the amount of string needed (in meters).

$$
x^{3}-4 x^{2}+5 x-\frac{19}{10}=0
$$

7.0 A cubic polynomial $f(x)=x^{3}+a x^{2}+b x+c$ with at least two distinct roots has the following properties:
(i) The sum of all the roots is equal to twice the product of all the roots.
(ii) The sum of the squares of all the roots is equal to 3 times the product of all the roots.
(iii) $f(1)=1$.

Find $c$. (The answer is 4 keys if you were to type it)
9.0 Let $a, b, c$ be numbers (imaginary or real) such that

$$
a+b+c=a b+a c+b c=a b c=1 .
$$

Find the actual values $a, b, c$, separated by semicolons.

## 5 Problems Level 3

10.0 The four zeros of the polynomial $x^{4}+j x^{2}+k x+225$ are distinct real numbers in arithmetic progression. Compute the value of $j$.
11.0 For some real number $r$, the polynomial $8 x^{3}-4 x^{2}-42 x+45$ is divisible by $(x-r)^{2}$. Find $r$.

The answer would be 3 keys if you were to type it! (Hint, don't be discouraged if you get a messy quadratic from your system of equations. Just use the quadratic formula and eliminate one *messier* solution for the answer)
12.0 The equation $x^{3}-8 x^{2}+8 x+2=0$ has three real roots, p,q,r. Find $\frac{1}{p^{2}}+\frac{1}{q^{2}}+\frac{1}{r^{2}}$. Surprisingly, the answer is an integer!
14.0 For certain real numbers $a, b$, and $c$, the polynomial

$$
g(x)=x^{3}+a x^{2}+x+10
$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$
f(x)=x^{4}+x^{3}+b x^{2}+100 x+c .
$$

What is $f(1)$ ? (The answer is negative, and is a palindrome!, if you exclude the negative that is).
15.0 The below polynomials have two roots in common; consequently, each cubic has a root that they do not have in common with the other cubic. Find the sum of the two roots that they do not have in common.

$$
\begin{aligned}
& x^{3}+5 x^{2}+p x+q=0 \\
& x^{3}+x^{2}+p x+r=0
\end{aligned}
$$

## 6 Bonus (You might not have time to solve these)

Let r be a complex number such that $r^{5}=1$ and r does not equal 1 . Compute $(r-1)\left(r^{2}-1\right)\left(r^{3}-1\right)\left(r^{4}-1\right)$

Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product which of the following numbers can be obtained?

Options: 22,60,119,194,231 (Only one answer is correct).

