MAT: Review: Matrices, Change of Bases, Vietas, Probability and Statistics Solutions

1. Add $4_6 + 14_6$. Express your answer in base 6.

When adding the numbers, we notice that $4 + 4$ leaves a residue of 2 when divided by 6. Thus, the sum will have a rightmost digit of 2, and we must carry-over. This yields that

$$\begin{array}{r} {}^{1}4_6 \\ +\ 14_6 \\ \hline 22_6 \end{array}$$

The sum is therefore $\boxed{22_6}$.

2. Find the difference between 1000_7 and 666_7 in base 7.

Setting up the subtraction and borrowing as shown:

$$\begin{array}{r} {}^{0\,6\,6\,7} \\ \cancel{1}\cancel{0}\cancel{0}\cancel{0}_7 \\ -66\,6_7 \\ \hline 1_7. \end{array}$$

So the difference is $\boxed{1_7}$.

3. Find the product of $218_9 \cdot 5_9$. Express your answer in base 9.

We begin by multiplying the units digit: $8_9 \times 5_9 = 40_{10} = 44_9$. So, we write down a 4 and carry-over another 4. Moving on to the next digit, we need to evaluate $1_9 \times 5_9 + 4_9 = 9_{10} = 10_9$. Thus, the next digit is a 0 and a 1 is carried over. Finally, the leftmost digits are given by the operation $2_9 \times 5_9 + 1_9 = 11_{10} = 12_9$. Writing this out, we have

$$\begin{array}{r} {}^{1\,4} \\ 218_9 \\ \times\quad 5_9 \\ \hline 1\,204_9 \end{array}$$

So our final answer is $\boxed{1204_9}$.

4. What is the positive difference between the probability of a fair coin landing heads up exactly 2 times out of 3 flips and the probability of a fair coin landing heads up 3 times out of 3 flips? Express your answer as a common fraction.

The probability that a fair coin lands heads up exactly 2 times out of 3 flips is $p_1 = \binom{3}{2}(1/2)^2(1/2) = 3/8$. The probability that a fair coin lands heads up 3 times out of 3 flips is $p_2 = (1/2)^3 = 1/8$. Finally, we have $p_1 - p_2 = 2/8 = \boxed{1/4}$.

5. The equations $x^3 + 5x^2 + px + q = 0$ and $x^3 + 7x^2 + px + r = 0$ have two roots in common. If the third root of each equation is represented by x_1 and x_2 respectively, compute the ordered pair (x_1, x_2).

If a is a root of both polynomials, then a is also a root of the difference of the polynomials, which is

$$(x^3 + 7x^2 + px + r) - (x^3 + 5x^2 + px + q) = 2x^2 + (r - q) = 0.$$

And if a is a root of this polynomial, so is $-a$, and their sum is 0.
By Vieta's formulas, the sum of the roots of $x^3 + 5x^2 + px + q = 0$ is -5, so the third root is -5. Similarly, the third root of $x^3 + 7x^2 + px + r = 0$ is -7, so $(x_1, x_2) = \boxed{(-5, -7)}$.

6. Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

We can substitute $x = y - 1$ to obtain a polynomial having roots $a + 1$, $b + 1$, $c + 1$, namely,

$$(y - 1)^3 - (y - 1) + 1 = y^3 - 3y^2 + 2y + 1.$$

The sum of the reciprocals of the roots of this polynomial is, by Vieta's formulas, $\frac{2}{-1} = \boxed{-2}$.

7. Compute $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^3$.

We have that

$$\begin{aligned} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^3 &= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}. \end{aligned}$$

8. Given $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, there exist positive real numbers x and y such that

$$(x\mathbf{I} + y\mathbf{A})^2 = \mathbf{A}.$$

Enter the ordered pair (x, y).

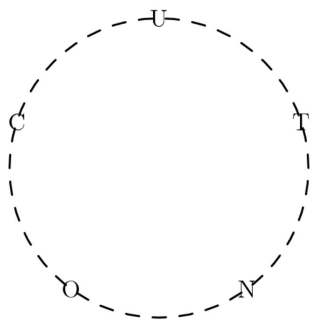
We have that

$$\begin{aligned} (x\mathbf{I} + y\mathbf{A}) &= \left(x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^2 \\ &= \begin{pmatrix} x & y \\ -y & x \end{pmatrix}^2 \\ &= \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ &= \begin{pmatrix} x^2 - y^2 & 2xy \\ -2xy & x^2 - y^2 \end{pmatrix}. \end{aligned}$$

We want this to equal $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so comparing coefficients, we get $x^2 - y^2 = 0$ and $2xy = 1$.

Then $x^2 = y^2$. Since x and y are positive, $x = y$. Then $2x^2 = 1$, or $x^2 = \frac{1}{2}$, so $(x, y) = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)}$.

9. The letters C, O, U, N and T are randomly placed around a circle. One such arrangement is shown here. If we always start with the C and continue to read the rest of the letters in order clockwise, in how many different orders can the letters appear?



Since we always read the letters clockwise, this is really the same as counting the number of linear permutations of the 5 letters, given that C must go first. So, there are $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 4! = \boxed{24}$ orders.

10. What is the base ten equivalent of 101010_5?

$101010_5 = 0 \cdot 5^0 + 1 \cdot 5^1 + 0 \cdot 5^2 + 1 \cdot 5^3 + 0 \cdot 5^4 + 1 \cdot 5^5 = 5 + 125 + 3125 = \boxed{3255}$