IB HL Mathematics: Counting and Probability

1. At a particular school with 43 students, each student takes chemistry, biology, or both. The chemistry class is three times as large as the biology class, and 5 students are taking both classes. How many people are in the chemistry class?

Solution: Let x be the number of students in the biology class who aren't in the chemistry class and y be the number of students in the chemistry class who aren't in the biology class. Then, since all students are in either one of the classes or in both, we know that $43 = x + y + 5$. We also know that $3(x + 5) = y + 5$. Solving for y in terms of x gives us $y = 3x + 10$, and substituting that into the first equation gives us $43 = x + (3x + 10) + 5$, which gives us $x = 7$. Substituting this into the other equation gives us $y = 31$. However, y is only the number of chemistry students who aren't taking biology, so we need to add the number of students taking both to get our final answer of $\boxed{36}$.

2. If all multiples of 3 and all multiples of 4 are removed from the list of whole numbers 1 through 100, then how many whole numbers are left?

Solutions:
We know that every third whole number starting from one must be removed from the list. Since the greatest multiple of 3 less than 100 is $3 \cdot 33 = 99$, this gives us a total of 33 such numbers. We then consider the multiples of four. Every fourth whole number starting from one is a multiple of four and since $4 \cdot 25 = 100$, this gives us 25 such numbers. However, we also have to account for the numbers that are multiples of both 3 and 4 which we counted twice. These are the multiples of 12 (the least common multiple of 3 and 4). Since $100 \div 12 = 8$ R4, we know that there are 8 multiples of both 3 and 4. Thus, we have $33 + 25 - 8 = 50$ numbers that we removed from the list. Since there were 100 whole numbers total, this leaves us with $100 - 50 = \boxed{50}$ whole numbers.

3. If a, b and c are three (not necessarily different) numbers chosen randomly and with replacement from the set $\{1, 2, 3, 4, 5\}$, what is the probability that $ab + c$ is even?

Solution: The quantity $ab + c$ is even if and only if ab and c are both odd or both even. The probability that c is odd is $\frac{3}{5}$, and the probability that ab is odd is $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$ (because both a and b must be odd). Therefore, the probability that $ab + c$ is even is

$$\frac{3}{5} \cdot \frac{9}{25} + \left(1 - \frac{3}{5}\right)\left(1 - \frac{9}{25}\right) = \boxed{\frac{59}{125}}.$$

4. If three, standard, 6-faced dice are rolled, what is the probability that the sum of the face up integers is 16?

Solution: At least one of the dice must come up 6, since otherwise the best we can do is 15. The other two dice must add up to 10. There are two ways two dice add to 10: $4+6$ and $5+5$.

So, we have two cases to consider:

A) The dice are 6, 6, 4. There are three possible ways this can happen, and the probability of each is $(1/6)^3 = 1/216$. So, the probability of this case is $3(1/216) = 1/72$.

B) The dice are 6, 5, 5. There are three possible ways this can happen, and the probability of each is $(1/6)^3 = 1/216$. So, the probability of this case is $3(1/216) = 1/72$.

Adding the probabilities of these two cases gives us a total probability of $\frac{1}{72} + \frac{1}{72} = \boxed{\frac{1}{36}}$.

5. A bag contains 10 red marbles and 6 blue marbles. Three marbles are selected at random and without replacement. What is the probability that one marble is red and two are blue? Express your answer as a common fraction.

Solution: There are three ways to draw two blue marbles and a red one: RBB, BRB, and BBR. Since there are no overlapping outcomes, these are distinct cases and their sum is the total probability that two of the three drawn will be blue. The desired probability therefore is

$$\frac{10}{16} \cdot \frac{6}{15} \cdot \frac{5}{14} + \frac{6}{16} \cdot \frac{10}{15} \cdot \frac{5}{14} + \frac{6}{16} \cdot \frac{5}{15} \cdot \frac{10}{14} = \boxed{\frac{15}{56}}.$$

Solution 2: There are $\binom{10}{1} = 10$ ways to choose a red marble, and $\binom{6}{2} = 15$ ways to choose two blue marbles. There are $\binom{16}{3} = 560$ ways to choose three marbles, so the desired probability is $\frac{10 \cdot 15}{560} = \boxed{\frac{15}{56}}$.

6. John rolls a pair of standard 6-sided dice. What is the probability that the two numbers he rolls are relatively prime? Express your answer as a common fraction.

Solution: Solution: We have to use a little bit of casework to solve this problem. If the first die shows a 1, the second die can be anything (6 cases). If the first die shows 2 or 4, the second die is limited to 1, 3, or 5 ($2 \cdot 3 = 6$ cases). If the first die shows 3, the second die can be 1, 2, 4, or 5 (4 cases). If the first die shows 5, the second die can be anything but 5 (5 cases). If the first die shows 6, the second die can be only 1 or 5 (2 cases). There are 36 ways to roll two dice, 23 of which are valid, so the answer is $\boxed{23/36}$.

7. A rectangle has a perimeter of 64 inches and each side has an integer length. How many non-congruent rectangles meet these criteria?

Solution: Call the height h and the width w. We want to find the number of solutions to $2(w+h)=64$ or $w+h=32$. The solutions to this are

$$\{(1,31),(2,30),\ldots,(16,16),\ldots,(31,1)\}.$$

There are 31 solutions to this, but we are double-counting all the rectangles for which $w \neq h$. There are 30 of these, so the total number of rectangles is $\frac{30}{2}+1=\boxed{16}$ rectangles.

8. a is chosen from the set $\{1,2,3,4\}$, and b is chosen from the set $\{5,6,7,8,9\}$. What is the probability that the product ab is a prime number?

Solution: The product ab is prime only when either a or b is one. Since b cannot equal 1, a must equal 1, and this occurs with probability $\frac{1}{4}$. Furthermore, b must be prime, so b must equal 5 or 7, and this occurs with probability $\frac{2}{5}$. Therefore, the probability that ab is prime is $\frac{1}{4}\cdot\frac{2}{5}=\boxed{\dfrac{1}{10}}$.

9. A bag has 4 red and 6 blue marbles. A marble is selected and not replaced, then a second is selected. What is the probability that both are the same color?

Solution: The probability that both marbles are red is given by:

$$P(\text{both red}) = P(\text{1st red}) \times P(\text{2nd red } \textbf{after} \text{ 1st red is drawn}).$$

The probability that the first marble is red is $\frac{4}{10}$. After drawing a red marble, there are 3 red marbles and 9 marbles total left in the bag, so the probability that the second marble is also red is $\frac{3}{9}$. Therefore

$$P(\text{both red}) = \frac{4}{10}\times\frac{3}{9}=\frac{2}{15}.$$

Similarly, the probability that both marbles are blue is given by:

$$P(\text{both blue}) = P(\text{1st blue}) \times P(\text{2nd blue } \textbf{after} \text{ 1st blue drawn}).$$

The probability that that the first marble is blue is $\frac{6}{10}$. After drawing a blue marble, there are 5 blue marbles and 9 marbles total left in the bag, so the probability that the second marble is also blue is $\frac{5}{9}$. Therefore

$$P(\text{both blue}) = \frac{6}{10}\times\frac{5}{9}=\frac{1}{3}.$$

Since drawing two red marbles and drawing two blue marbles are exclusive events, we add the individual probabilities to get the probability of one or the other occurring. Therefore:

$$\begin{aligned} P(\text{both same color}) &= P(\text{both red}) + P(\text{both blue}) \\ &= \frac{2}{15}+\frac{1}{3}=\boxed{\frac{7}{15}}. \end{aligned}$$

10. When Trilisa takes pictures, they turn out with probability $\frac{1}{5}$. She wants to take enough pictures so that the probability of at least one turning out is at least $\frac{3}{4}$. How few pictures can she take to accomplish this?

Solution: The probability that at least one picture turns out is 1 minus the probability that all the pictures do not turn out. Since the probability that one picture will not turn out is $\frac{4}{5}$, the probability that n pictures all do not turn out is $\left(\frac{4}{5}\right)^n$. So we want

$$\left(\frac{4}{5}\right)^n < \frac{1}{4} \Rightarrow 4^{n+1} < 5^n$$

We see that $4^7 > 5^6$, but $4^8 < 5^7$. Thus the smallest allowable value of n is $\boxed{7}$.