

# Mu Alpha Theta Precalculus: Matrices Solutions

1. Find the  $2 \times 2$  matrix  $\mathbf{M}$  such that  $\mathbf{M} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -15 \\ -6 \end{pmatrix}$  and  $\mathbf{M} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ 18 \end{pmatrix}$ .

Let  $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then

$$\mathbf{M} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 2a + 7b \\ 2c + 7d \end{pmatrix}.$$

Also,

$$\mathbf{M} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4a - b \\ 4c - d \end{pmatrix}.$$

Thus, we have the system of equations

$$\begin{aligned} 2a + 7b &= -15, \\ 2c + 7d &= -6, \\ 4a - b &= 15, \\ 4c - d &= 18. \end{aligned}$$

Solving this system, we find  $a = 3$ ,  $b = -3$ ,  $c = 4$ , and  $d = -2$ , so

$$\mathbf{M} = \boxed{\begin{pmatrix} 3 & -3 \\ 4 & -2 \end{pmatrix}}.$$

2. Compute

$$\begin{pmatrix} 3a^2 - 3 & 3a \\ 2a^2 - a - 2 & 2a - 1 \end{pmatrix} \begin{pmatrix} -1 & -3a - 2 \\ a & 3a^2 + 2a - 3 \end{pmatrix}.$$

We compute

$$\begin{aligned} &\begin{pmatrix} 3a^2 - 3 & 3a \\ 2a^2 - a - 2 & 2a - 1 \end{pmatrix} \begin{pmatrix} -1 & -3a - 2 \\ a & 3a^2 + 2a - 3 \end{pmatrix} \\ &= \begin{pmatrix} (3a^2 - 1)(-1) + (3a)(a) & (3a^2 - 3)(-3a - 2) + (3a)(3a^2 + 2a - 3) \\ (2a^2 - a - 2)(-1) + (2a - 1)(a) & (2a^2 - a - 2)(-3a - 2) + (2a - 1)(3a^2 + 2a - 3) \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}}. \end{aligned}$$

3. Find  $\begin{pmatrix} 1 & 5 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -3 \\ 8 & -5 \end{pmatrix}$ .

We have that

$$\begin{pmatrix} 1 & 5 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -3 \\ 8 & -5 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 \\ 6 & -1 \end{pmatrix}}.$$

4. Find the  $2 \times 2$  matrix  $\mathbf{M}$  such that

$$\mathbf{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

and

$$\mathbf{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}.$$

In general,  $\mathbf{M} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is the first column of  $\mathbf{M}$ , and  $\mathbf{M} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is the second column of  $\mathbf{M}$ , so

$$\mathbf{M} = \boxed{\begin{pmatrix} 3 & 2 \\ 0 & -7 \end{pmatrix}}.$$

5. Let  $\mathbf{M} = \begin{pmatrix} 0 & -5 \\ -2 & 4 \end{pmatrix}$ . There exist scalars  $p$  and  $q$  such that

$$\mathbf{M}^2 = p\mathbf{M} + q\mathbf{I}.$$

Enter the ordered pair  $(p, q)$ .

Since  $\mathbf{M}^2 = \begin{pmatrix} 0 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 0 & -5 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 10 & -20 \\ -8 & 26 \end{pmatrix}$ , we seek  $p$  and  $q$  such that

$$\begin{pmatrix} 10 & -20 \\ -8 & 26 \end{pmatrix} = p \begin{pmatrix} 0 & -5 \\ -2 & 4 \end{pmatrix} + q \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we want  $p$  and  $q$  to satisfy  $q = 10$ ,  $-5p = -20$ ,  $-2p = -8$ , and  $4p + q = 26$ . Solving, we find  $(p, q) = \boxed{(4, 10)}$ .

6. Find

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 99 \\ 0 & 1 \end{pmatrix}.$$

More generally,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 99 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+3+5+\cdots+99 \\ 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2500 \\ 0 & 1 \end{pmatrix}}.$$