MAT: Logarithms Solutions

1. What is $\log_7 2400$ rounded to the nearest integer?

We can have $\log_7 343 = 3$ and $\log_7 2401 = 4$. Since $\log_7 x$ increases as x increases, we know that $\log_7 343 < \log_7 2400 < \log_7 2401$, meaning $3 < \log_7 2400 < 4$. Moreover, we can see that 2400 is much closer to 2401 than to 343, so it stands to reason that $\log_7 2400$ rounded to the nearest integer is $\boxed{4}$.

2. Evaluate $\log_4 32$.

Let $x = \log_4 32$. Then, we must have $4^x = 32$. Writing both 4 and 32 with 2 as the base gives $(2^2)^x = 2^5$, so $2^{2x} = 2^5$. Therefore, we must have $2x = 5$, so $x = \boxed{5/2}$.

3. If r, s, and t are constants such that $\frac{x^{r-2}\cdot y^{2s}\cdot z^{3t+1}}{x^{2r}\cdot y^{s-4}\cdot z^{2t-3}} = xyz$ for all non-zero x, y, and z, then solve for $r^s \cdot t$. Express your answer as a fraction.

First, we should solve for r, s, and t. From what is given, we know that $\frac{x^{r-2}}{x^{2r}} = x$, $\frac{y^{2s}}{y^{s-4}} = y$, and $\frac{z^{3t+1}}{z^{2t-3}} = z$. Solving for r, s, and t we have:

$$\begin{aligned} r - 2 = 2r + 1 &\Rightarrow r = -3 \\ 2s = s - 4 + 1 &\Rightarrow s = -3 \\ 3t + 1 = 2t - 3 + 1 &\Rightarrow t = -3 \end{aligned}$$

Solving for $r^s \cdot t$, we have $(-3)^{-3} \cdot -3 = \frac{-1}{27} \cdot -3 = \boxed{\frac{1}{9}}$.

4. How many distinct four-tuples (a, b, c, d) of rational numbers are there with

$$a \cdot \log_{10} 2 + b \cdot \log_{10} 3 + c \cdot \log_{10} 5 + d \cdot \log_{10} 7 = 2005?$$

We can write the given equation as

$$\log_{10} 2^a + \log_{10} 3^b + \log_{10} 5^c + \log_{10} 7^d = 2005.$$

Then

$$\log_{10}(2^a \cdot 3^b \cdot 5^c \cdot 7^d) = 2005,$$

so $2^a \cdot 3^b \cdot 5^c \cdot 7^d = 10^{2005}$.
Since a, b, c, d are all rational, there exists some positive integer M so that aM, bM, cM, dM are all integers. Then

$$2^{aM} \cdot 3^{bM} \cdot 5^{cM} \cdot 7^{dM} = 10^{2005M} = 2^{2005M} \cdot 5^{2005M}.$$

From unique factorization, we must have $aM = 2005M$, $bM = 0$, $cM = 2005M$, and $dM = 0$. Then $a = 2005$, $b = 0$, $c = 2005$, and $d = 0$. Thus, there is only $\boxed{1}$ quadruple, namely $(a, b, c, d) = (2005, 0, 2005, 0)$.

5. Solve for the rational numbers x and y:

$$2^{x+y} \cdot 3^{x-y} \cdot 6^{2x+2y} = 72.$$

Express your answer as an ordered pair (x, y).

Combining the bases on the left side, we obtain

$$2^{x+y}3^{x-y}2^{2x+2y}3^{2x+2y} = 2^{3x+3y}3^{3x+y} = 2^3 3^2.$$

Then, by setting exponents equal to each other, we obtain the simultaneous equations

$$\begin{aligned} 3x + 3y &= 3 \\ 3x + y &= 2. \end{aligned}$$

Subtracting the second equation from the first, we obtain $2y = 1 \implies y = \frac{1}{2}$, and plugging $y = \frac{1}{2}$ back in, we get $x = \frac{1}{2}$. So, $(x, y) = \boxed{(1/2, 1/2)}$.

6. How many integers $-11 \leq n \leq 11$ satisfy $(n-2)(n+4)(n+8) < 0$?

Since $(n-2)(n+4)(n+8) = 0$ when $n = 2, -4$, or -8, we will consider the four cases $-11 \leq n < -8$, $-8 < n < -4$, $-4 < n < 2$, and $2 < n \leq 11$ separately. If $n = 2$, $n = -4$, or $n = -8$, then all three factors are 0. If $n > 2$, then all three factors are positive. If $-4 < n < 2$, then $n - 2$ is negative, while the other two factors are positive, so the product is negative. If $-8 < n < -4$, then $n + 8$ is positive, while the other two factors are negative, so the product is positive. If $n < -8$, then all three factors are negative, so the product is negative. In total, there are $\boxed{8}$ solutions: $-11, -10, -9, -3, -2, -1, 0, 1$.

7. Find the sum of all the solutions to

$$(3\log_4(\log_3 x))(2\log_{64}(\log_3 x) - 1) = -1.$$

Let $y = \log_3 x$. Then the equation becomes

$$(3\log_4 y)(2\log_{64} y - 1) = -1.$$

By the change-of-base formula,

$$\log_{64} y = \frac{\log_4 y}{\log_4 64} = \frac{\log_4 y}{3},$$

so

$$(3\log_4 y)\left(\frac{2}{3} \cdot \log_4 y - 1\right) = -1.$$

Let $z = \log_4 y$, so $(3z)\left(\frac{2}{3}z - 1\right) = -1$. This simplifies to $2z^2 - 3z + 1 = 0$, which factors as $(z-1)(2z-1) = 0$. The solutions are $z = 1$ and $z = \frac{1}{2}$. Then the possible values of y are $4^1 = 4$ and $4^{1/2} = 2$, and the possible values of x are $3^4 = 81$ and $3^2 = 9$. The sum of the solutions is then $81 + 9 = \boxed{90}$.

8. Let x, y, and z be real numbers such that

$$\begin{aligned} \log_2(xyz - 3 + \log_5 x) &= 5, \\ \log_3(xyz - 3 + \log_5 y) &= 4, \\ \log_4(xyz - 3 + \log_5 z) &= 4. \end{aligned}$$

Find xyz.

From the given equations,

$$\begin{aligned}
xyz - 3 + \log_5 x &= 32, \\
xyz - 3 + \log_5 y &= 81, \\
xyz - 3 + \log_5 z &= 256.
\end{aligned}$$

Adding these equations, we get

$$3xyz - 9 + \log_5 x + \log_5 y + \log_5 z = 369,$$

which turns into

$$3xyz + \log_5(xyz) = 378.$$

Consider the function

$$f(t) = 3t + \log_5 t.$$

Since both $3t$ and $\log_5 t$ are increasing functions, the function $f(t)$ is also increasing. Furthermore,

$$f(125) = 3 \cdot 125 + \log_5 125 = 378,$$

so $t = 125$ is the unique solution to $3t + \log_5 t = 378$. Hence, $xyz = \boxed{125}$.

With some more work, we can show that $x = \frac{1}{5^{90}}$, $y = \frac{1}{5^{41}}$, and $z = 5^{134}$.

9. Let $f(x) = \log_b x$, and let $g(x) = x^2 - 4x + 4$. given that $f(g(x)) = g(f(x)) = 0$ has exactly one solution and that $b > 1$, compute b.

We can write $g(x) = x^2 - 4x + 4 = (x-2)^2$. Thus,

$$\log_b[(x-2)^2] = (\log_b x - 2)^2 = 0.$$

From the equation $\log_b[(x-2)^2] = 0$, $(x-2)^2 = 1$, so $x = 1$ or $x = 3$.

From the equation $(\log_b x - 2)^2 = 0$, $\log_b x = 2$, so $x = b^2$.

The only possibility is then $x = 3$ and $b = \boxed{\sqrt{3}}$.

10. Let x,y, and z all exceed 1, and let w be a positive number such that $\log_x w = 24$, $\log_y w = 40$, and $\log_{xyz} w = 12$. Find $\log_z w$.

We have the identity $\log_b a = \frac{1}{\log_a b}$, so

$$\log_w x = \frac{1}{24}, \quad \log_w y = \frac{1}{40}, \quad \text{and} \quad \log_w(xyz) = \frac{1}{12}.$$

Now, since

$$\log_w(xyz) = \log_w x + \log_w y + \log_w z,$$

we have

$$\frac{1}{12} = \frac{1}{24} + \frac{1}{40} + \log_w z,$$

so $\log_w z = \frac{1}{12} - \frac{1}{24} - \frac{1}{40} = \frac{1}{60}$. Then

$$\log_z w = \frac{1}{\log_w z} = \boxed{60}.$$