MAT: General Geometry Solutions

1. What is the number of square units in the area of a triangle whose sides measure 5, 5 and 6 units?

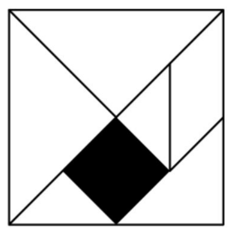
Using Heron's formula, we have $s = \dfrac{5+6+6}{2} = 8$. So then we have

$$A = \sqrt{8(8-5)(8-5)(8-6)} = 12$$

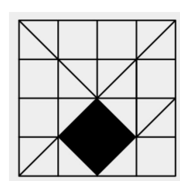
2. A rhombus has an area of 108 square units. The lengths of its diagonals have a ratio of 3 to 2. What is the length of the longest diagonal, in units?

Let the diagonals have length $3x$ and $2x$. Half the product of the diagonals of a rhombus is equal to the area, so $(2x)(3x)/2 = 108$. Solving for x, we find $x = 6$. Therefore, the length of the longest diagonal is $3x = \boxed{18}$.

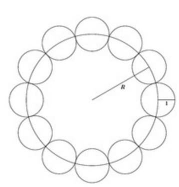
3. What is the ratio of the area of the shaded square to the area of the large square? (The figure is drawn to scale.)



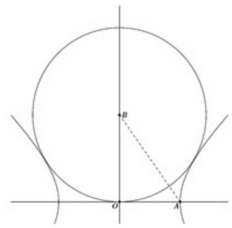
Divide the square into 16 smaller squares as shown. The shaded square is formed from 4 half-squares, so its area is 2. The ratio 2 to 16 is $\boxed{\dfrac{1}{8}}$.



4. Twelve circles with radius 1 are arranged so that the centers lie on a large circle and each circle is tangent to its 2 neighbors, as in the diagram. If R is the radius of the large circle, then R^2 can be expressed in simplest form as $a + b\sqrt{c}$, where the prime factorization of c has no squares. What is $a + b + c$?



5. Circle B is tangent to the x-axis and to both branches of the hyperbola $x^2 - ay^2 = m^2$, as shown in the diagram. If $OB : OA = 2 : 1$, find the value of a.



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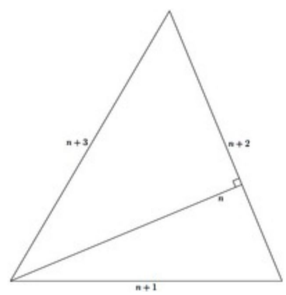
6. A triangle has sides which are consecutive integers n+1, n+2, and n+3 and an altitude of length n with its foot on the side of length n+2. The integers are the least possible. What is the perimeter of the triangle?



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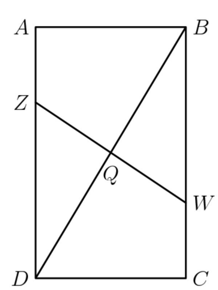
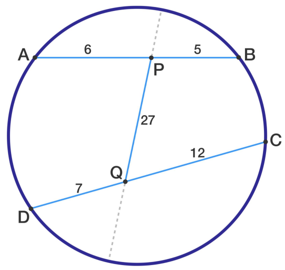
7. In the figure, $ABCD$ is a rectangle, $AZ = WC = 6$ units, $AB = 12$ units and the area of trapezoid $ZWCD$ is 120 square units. What is the area of triangle BQW?



Because the figure has rotational symmetry, Q is the midpoint of ZW. Consequently, the triangles BZQ and BWQ have the same area because they share a height and have bases that are the same length. We have

$$[BQW] = \frac{1}{2}[BZW] = \frac{1}{2}([ABWZ] - [ABZ])$$
$$= \frac{1}{2}\left(120 - \frac{1}{2} \cdot 6 \cdot 12\right) = \frac{1}{2}(120 - 36) = \frac{84}{2} = \boxed{42}.$$

8. $ABCD$ is a cyclic quadrilateral with $\overline{AB} = 11$ and $\overline{CD} = 19$. P and Q are points on $\overline{AB}$ and $\overline{CD}$, respectively, such that $\overline{AP} = 6$, $\overline{DQ} = 7$, and $\overline{PQ} = 27$. Determine the length of the line segment formed when $\overline{PQ}$ is extended from both sides until it reaches the circle.



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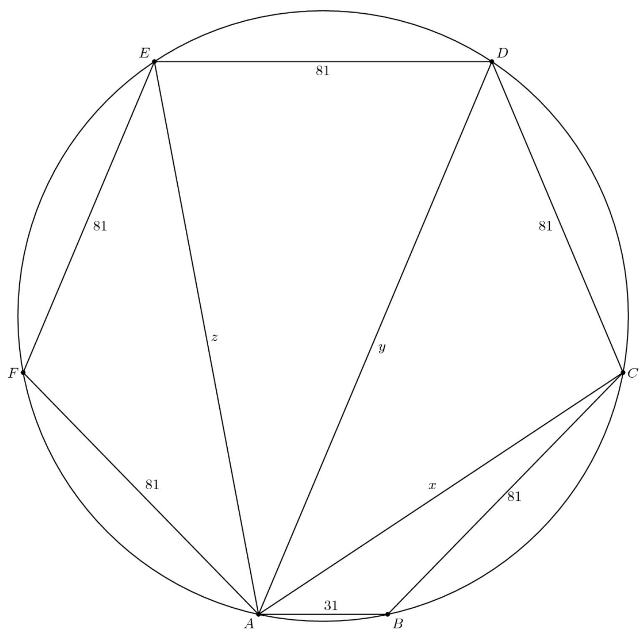
9. In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of AD/CD?

Set $\overline{BD}$'s length as x. CD's length must also be x since $\angle BAD$ and $\angle DAC$ intercept arcs of equal length(because $\angle BAD = \angle DAC$). Using Ptolemy's theorem, $7x + 8x = 9(AD)$. The ratio is $\boxed{\frac{5}{3}}$

10. A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.



Let $x = AC = BF$, $y = AD = BE$, and $z = AE = BD$.
Ptolemy's Theorem on $ABCD$ gives $81y + 31 \cdot 81 = xz$, and Ptolemy on $ACDF$ gives $x \cdot z + 81^2 = y^2$. Subtracting these equations give $y^2 - 81y - 112 \cdot 81 = 0$, and from this $y = 144$. Ptolemy on $ADEF$ gives $81y + 81^2 = z^2$, and from this $z = 135$. Finally, plugging back into the first equation gives $x = 105$, so $x + y + z = 105 + 144 + 135 = \boxed{384}$.